If z is less than this, y is less than $\frac{a}{6}\sqrt{3}(1+\sin\theta \operatorname{cosec}\phi)$.

If $z>\frac{a}{3}(1+\sin\phi \operatorname{cosec}\theta)$ and $y<\frac{a}{6}\sqrt{3}(1+\sin\theta \operatorname{cosec}\phi)$ the limits of z are $\frac{a}{3}(1+\sin\phi \operatorname{cosec}\theta)$ and a.

$$\varDelta=\triangle\Big[\int_0^{\frac{1}{6}\pi}\tfrac{1}{32}\sin\theta \operatorname{cosec}^3\phi-\tfrac{1}{48}\sin^2\theta \operatorname{cosec}^4\phi+\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi}\tfrac{1}{48}\sin^2\theta \operatorname{cosec}^4\phi-\frac{1}{2\cdot3^7}$$
$$\times(\sin^2\theta \operatorname{cosec}^4\phi+\sin^2\phi \operatorname{cosec}^2\theta)+\frac{1}{2^5\cdot3^6}(\sin\theta \operatorname{cosec}^3\phi+\sin\phi \operatorname{cosec}^3\theta+\frac{41}{2^4,3^7}$$
$$\times \operatorname{cosec}\theta \operatorname{cosec}\phi\Big]d\theta\div\tfrac{1}{32}\int_0^{\frac{1}{3}\pi}\sin\theta \operatorname{cosec}^3\phi\, d\theta=\frac{\triangle}{3^6}(470+\tfrac{41}{3}\log 2)=.6997\triangle.$$

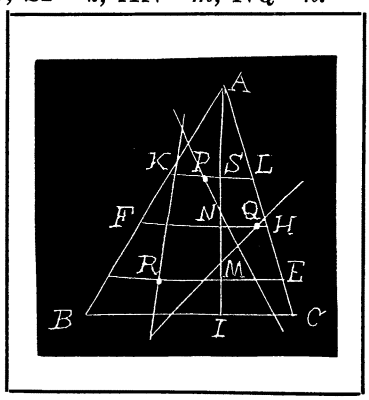
This is problem 76, p. 513, Williamson's *Integral Calculus.*

170. Proposed by LON C. WALKER, A. M., Santa Barbara, California.

Find the area of a triangle formed by drawing a line at random through each of three points taken at random within a given triangle.

Solution by G. B. M. ZERR, Ph. D., Parsons, W. Va.

Let ABC be the given triangle; P, Q, R the random points; $AI=h$, $BI=d$, $CI=e$, $d+e=a$, $AM=u$, $MR=v$, $AS=w$, $SP=z$, $AN=m$, $NQ=n$.



$y-v=r(x-u)$, the line through R...(1),
$y-z=s(x-w)$, the line through P...(2),
$y-n=t(x-m)$, the line through Q...(3),
where $r=\tan\theta$, $s=\tan\phi$, $t=\tan\psi$.

The intersection of (1) and (2) is given by

$$x_1=\frac{ru-ws+z-v}{r-s},\quad y_1=\frac{rsu-rsw+rz-sv}{r-s}.$$

The intersection of (1) and (3) is given by

$$x_2=\frac{ru-mt+n-v}{r-t},\quad y_2=\frac{rtu-mrt+rn-tv}{r-t}.$$

The intersection of (2) and (3) is given by

$$x_3=\frac{sw-mt+n-z}{s-t},\quad y_3=\frac{stw-mst+sn-tz}{s-t}.$$

Area of triangle $=\frac{1}{2}(x_2y_1-x_1y_2+x_3y_2-x_2y_3+x_1y_3-x_3y_1)$
$=\frac{1}{2}[(ur-v)(s-t)+(ws-z)(t-r)+(mt-n)(r-s)]^2/(r-s)(r-t)(s-t)$
$=A/B$.

The limits of u are 0 and h; of w, 0 and u; of m, w and u; of v, $-du/h=v_1$ and $eu/h=v_2$; of z, $-dw/h=w_1$ and $ew/h=w_2$; of n, $-dm/h=m_1$ and $em/h=m_2$. The number of ways the three points can be taken on the surface of the triangle is $\frac{1}{6}(\frac{1}{2}ah)^3=\frac{1}{48}a^3h^3$.

Hence if $\triangle$ =the required average area we get

$$\triangle=\frac{48}{a^3h^3}\int_0^h\int_0^u\int_w^u\int_{v_1}^{v_2}\int_{w_1}^{w_2}\int_{n_1}^{n_2}\frac{A}{B}du\,dw\,dm\,dv\,dz\,dn$$

$$=\frac{8}{ah^8B}\int_0^h\int_0^u\int_w^u\{(s-t)^2[3ah^2r^2+3hr(d^2-e^2)+d^3+e^3]u^3wm$$

$$+(t-r)^2[3ah^2s^2+3hs(d^2-e^2)+d^3+e^3]uw^3m$$

$$+(r-s)^3[3ah^2t^2+3ht(d^2-e^2)+d^3+e^3]uwm^3\}du\,dw\,dm$$

$$+\frac{12}{a^2h^8B}\int_0^h\int_0^u\int_w^u[(s-t)(t-r)(2ahr+d^2-e^2)(2ahs+d^2-e^2)u^2w^2m$$

$$+(s-t)(r-s)(2ahr+d^2-e^2)(2aht+d^2-e^2)u^2wm^2$$

$$+(t-r)(r-s)(2aht+d^2-e^2)(2ahs+d^2-e^2)uw^2m^2]du\,dw\,dm$$

$$=\frac{1}{24a}\Big[\frac{9ah^2r^2(s-t)}{(r-s)(r-t)}+\frac{9hr(s-t)(d^2-e^2)}{(r-s)(r-t)}+\frac{3(d^3+e^3)(s-t)}{(r-s)(r-t)}$$

$$+\frac{3ah^2s^2(r-t)}{(r-s)(s-t)}+\frac{3hs(r-t)(d^2-e^2)}{(r--s)(s-t)}+\frac{(d^3+e^3)(r-t)}{(r-s)(s--t)}$$

$$+\frac{6ah^2t^2(r-s)}{(s-t)(r-t)}+\frac{6ht(r-s)(d^2-e^2)}{(s-t)(r-t)}+\frac{2(d^3+e^3)(r-s)}{(s-t)(r-t)}\Big]$$

$$+\frac{1}{60a^2}\Big[\frac{24a^2h^2rs}{s-r}+\frac{36a^2h^2rt}{r-t}+\frac{20a^2h^2st}{t-s}+\frac{12ah(d^2-e^2)(r+s)}{s-r}$$

$$+\frac{18ah(d^2-e^2)(r+t)}{r-t}+\frac{10ah(d^2-e^2)(s+t)}{t-s}+\frac{6(d^2-e^2)}{s-r}+\frac{9(d^2-e^2)^2}{r-t}$$

$$+\frac{5(d^2-e^2)^2}{t-s}\Big]=\frac{1}{24a}C+\frac{1}{60a^2}D.$$

The limits of θ, ϕ, and ψ are for each 0 and $\frac{1}{2}\pi$ and doubled.

Now $\frac{r^2(s-t)}{(r-s)(r-t)}=\frac{r^2}{r-s}-\frac{r^2}{r-t}$.

$$\therefore \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{r^2(s-t)}{(r-s)\ (r-t)}d\theta\, d\phi\, d\psi=\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{s^2(r-t)}{(r-s)\ (s-t)}d\theta d\phi d\psi$$

$$=\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{t^2(r-s)}{(s-t)\ (r-t)}d\theta\, d\phi\, d\psi=0.$$

$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{r(s-t)}{(r-s)\ (r-t)}d\theta\, d\phi\, d\psi=\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{s(r-t)}{(r-s)\ (s-t)}d\theta\, d\phi\, d\psi$$

$$=\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{t(r-s)}{(s-t)\ (r-t)}d\theta\, d\phi\, d\psi=0.$$

$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{(s-t)}{(r-s)\ (r-t)}d\theta\, d\phi\, d\psi=\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{(r-t)}{(r-s)\ (s-t)}d\theta\, d\phi\, d\psi$$

$$=\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{(r-s)}{(s-t)\ (r-t)}d\theta\, d\phi\, d\psi=0. \quad \therefore C=0.$$

$$\int_0^{\frac{1}{3}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{rs}{r-s}d\theta\, d\phi\, d\psi=\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{rt}{r-t}d\theta\, d\phi\, d\psi$$

$$=\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{st}{t-s}d\theta\, d\phi\, d\psi=\tfrac{1}{8}\pi^2.$$

$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{r+s}{r-s}d\theta\, d\phi\, d\psi=\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{r+t}{r-t}d\theta\, d\phi\, d\psi$$

$$=\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{s+t}{t-s}d\theta\, d\phi\, d\psi=0.$$

$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{d\theta\, d\phi\, d\psi}{s-r}=\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{d\theta\, d\phi\, d\psi}{r-t}=\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{d\theta\, d\phi\, d\psi}{t-s}=-\tfrac{1}{8}\pi^2.$$

$$\therefore D=\frac{10\pi^2a^2h^2-\frac{5}{2}\pi^2(d^2-e^2)^2}{\frac{1}{8}\pi^3}.$$

$$\therefore \triangle=\frac{4\,a^2h^2-(d^2-e^2)^2}{3\pi a^2}=\frac{[a^2(2c^2+2b^2-a^2)\ -\ 2(c^2-b^2)^2]}{3\pi a^2}.$$

If $a=b=c$, $\triangle=a^2/\pi$.